PARADOX

Galileo's kinematical paradox and the expanding circle of simultaneity

Consider a vertical circle and the diameter *BA* and representative chords *BC*, *BD*, which start from point *B* at the top of the circle, and also the chord *EA* and the line segment *GF* (see figure 1). If we simultaneously release two or more beads from *B* in such a way that one of the beads falls along diameter *BA* and the others slide along chords *BC* and *BD* or *EA*, then the beads will hit the circumference of the circle at the same instant, even though the chords have different lengths [1, 2]. This apparent paradox can easily be explained if we relate the length of diameter *BA*, let us call it h, to the length of a chord, say *EA*, let us call it ℓ. Then, by making use of Thales' theorem, which states that any triangle inscribed in a semicircle is a right triangle, we can write

$$\frac{\ell}{h} = \frac{|EA|}{|BA|} = \sin\theta, \tag{1}$$

where θ is the measure of the angle between chord *EA* and horizontal line *GF*. It also follows that the acceleration along the diameter and the chord are related in the same way

$$\frac{a}{g} = \sin\theta. \tag{2}$$

With this result, we can easily explain the paradox as follows. Consider two beads that start from rest simultaneously, one from the top of the vertical circle *B* and the other from the extremity *E* of chord *EA*. For these two bodies, we can write

$$h = \frac{1}{2} g\, t_h^2, \tag{3}$$

and

$$\ell = \frac{1}{2} a\, t_\ell^2, \tag{4}$$

where t_h and t_ℓ are the duration of the free fall along the diameter and of the slide along the chord, respectively. If we take equations (1) and (2) into equation (4) we reobtain equation (3), that is, we will have shown that $t_h = t_\ell$.

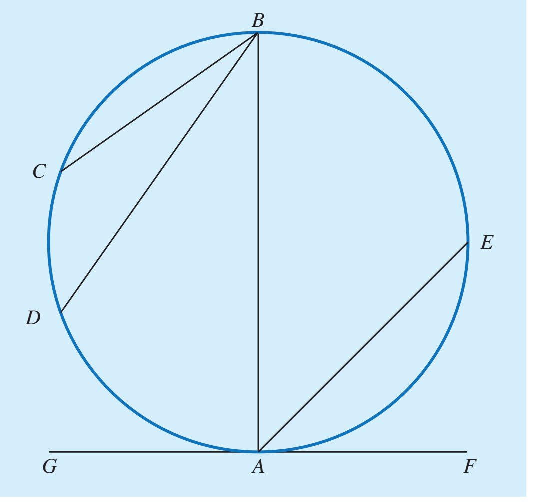

Figure 1. *A vertical circle and some of its chords.*

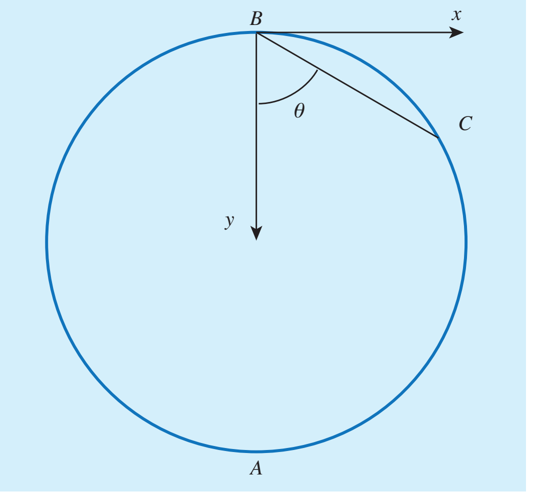

Figure 2. *The reference system used to describe the expanding circle.*

The same procedure can be employed to show that if the two beads are released simultaneously from the top of the vertical circle and one of them falls freely along the diameter and the other slides along chord *BE*, then both will also arrive at the circumference at the same time. In fact, we can

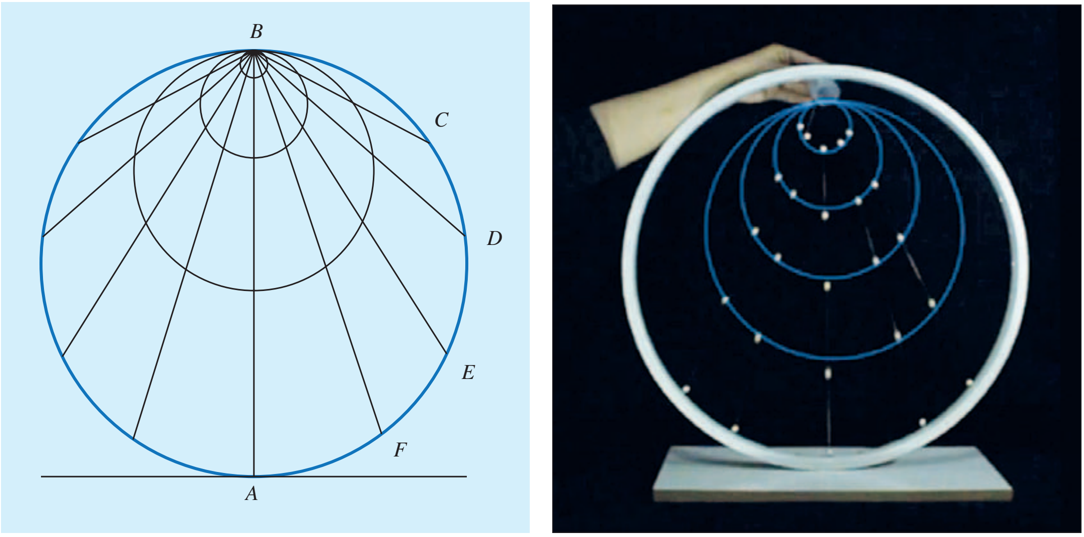

Figure 3. *The expanding circle of simultaneity and its realization as a superposition of five frames.*

choose any chord that we want. Two short video demonstrations can be used to illustrate the apparent paradox [3, 4].

But there is more to it. For an arbitrary t, the beads released simultaneously from point B are located on a circumference whose centre moves along BA, while its radius increases with time until it coincides with the circumference of the vertical circle. Next we describe this effect analytically.

The expanding circle of simultaneity

Consider the vertical circle shown in figure 2. Let g be the free-fall acceleration along diameter BA and a the acceleration along chord BC. Set the origin of the reference system at the top of the vertical circle with the Cartesian axes oriented as shown in figure 2 and consider the projections of the acceleration a along the axes x and y

$$a_x = a\cos(90°-\theta) = g\cos\theta\sin\theta = \frac{g}{2}\sin(2\theta), \tag{5}$$

and

$$a_y = a\cos\theta = g\cos^2\theta. \tag{6}$$

Then the coordinates x and y as functions of time are given by

$$x = \frac{1}{2}a_x t^2 = \frac{1}{2}\frac{g}{2}t^2\sin(2\theta), \tag{7}$$

and

$$y = \frac{1}{2}a_y t^2 = \frac{1}{2}g\,t^2\cos^2\theta. \tag{8}$$

Making use of the following trigonometric relation

$$\cos(2\theta) = 2\cos^2 - 1, \tag{9}$$

we can write

$$y = \frac{1}{2}\frac{g}{2}t^2[1+\cos(2\theta)]. \tag{10}$$

Combining equations (7) and (10) it follows that

$$x^2 + \left(y - \frac{1}{2}\frac{g}{2}t^2\right)^2 = \left(\frac{1}{2}\frac{g}{2}t^2\right)^2. \tag{11}$$

This equation represents a circle with a time-dependent radius given by

$$R(t) = \frac{1}{2}\frac{g}{2}t^2, \tag{12}$$

whose centre falls vertically along the diameter with an acceleration equal to half the free-fall acceleration. Figure 3 shows some representative circles.

This also means that if we draw chords that have point A at the top of the vertical circle as a common point, and from this point we simultaneously release from rest several beads at a given moment *t*, then these beads will be located on a circle whose radius is given by equation (12). A video demonstration of the expanding circle of simultaneity is available [5]. In figure 3, we show an image taken from the video where we have superimposed five frames to illustrate the phenomenon.

Final remarks

A simple discussion of Galileo's kinematical paradox along the lines we have followed here is given in [6]. Galileo's expanding circle of simultaneity can also be discussed in a purely geometrical way as in the classic book by Jeans [7], where there is a brief discussion.

A demonstration setup concerning Galileo's kinematical paradoxes is described in Sutton [8]. The three videos cited in the references were shot using a simple camera, and the demo setup was built with low-cost materials, a repainted used bicycle wheel, segments of metallic and nylon wires, and plastic beads. Our experience has shown that although the videos are unsophisticated and have limitations—for instance it is physically impossible to simultaneously release four beads from the same point—they are capable of motivating most students. The students find the (apparent) paradoxes intriguing and this can be a good starting point towards presenting free-fall and constrained motions.

References

[1] del Monte G *Epistolario Guidobaldo del Monte* www.urbinoelaprospettiva.it/Epistolario GuidUbaldo.pdf

[2] Galilei G 1954 *Dialogues Concerning Two New Sciences* (New York: Dover) (tr. H Crew and A de Salvio). See also: Galilei G 1989 *Two New Sciences* 2nd edn (Toronto: Wall and Emerson)

[3] www.youtube.com/watch?v=tUnhCPGsJxw&feature=youtu.be

[4] www.youtube.com/watch?v=HRtjvm2pVm0

[5] www.youtube.com/watch?v=eqWQNMgk7i0

[6] Greenslade Jr T B 2008 Galileo's paradox *Phys. Teach.* **46** 294

[7] Jeans J H 1967 *An Elementary Treatise on Theoretical Mechanics* (New York: Dover)

[8] Sutton R M 1938 *Demonstration Experiments in Physics* (London: McGraw-Hill) pp 42–3

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